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Some new Fourier inequalities for unbounded orthogonal systems in Lorentz–Zygmund spaces

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Abstract

In this paper we prove some essential complements of the paper (J. Inequal. Appl. 2019:171, 2019) on the same theme. We prove some new Fourier inequalities in the case of the Lorentz–Zygmund function spaces $L_{q,r}(\log L)^\alpha$ involved and in the case with an unbounded orthonormal system. More exactly, in this paper we prove and discuss some new Fourier inequalities of this type for the limit case $L_{2,r}(\log L)^\alpha$, which could not be proved with the techniques used in the paper (J. Inequal. Appl. 2019:171, 2019).

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1 Introduction

Let $q \in (1, +\infty)$, $r \in (0, +\infty)$ and $\alpha \in \mathbb{R}$. As usual $L_{q,r}(\log L)^\alpha$ denotes the Lorentz–Zygmund space, which consists of all measurable functions f on $[0, 1]$ such that

$$\|f\|_{q,r,\alpha} := \left\{ \int_0^1 (f^*(t))^r (1 + |\ln t|)^{\alpha r} \cdot t^{\frac{r}{q}-1} dt \right\}^{\frac{1}{r}} < +\infty,$$

where f^* denotes a nonincreasing rearrangement of the function $|f|$ (see e.g. [21]).

If $\alpha = 0$, then the Lorentz–Zygmund space coincides with the Lorentz space $L_{q,r}(\log L)^\alpha = L_{q,r}$ so that, if in addition $r = q$, then $L_{q,r}(\log L)^\alpha$ space coincides with the Lebesgue space $L_q[0, 1]$ (see e.g. [14]) with the norm

$$\|f\|_q := \left(\int_0^1 |f(x)|^q dx \right)^{\frac{1}{q}}, \quad 1 \leq q < +\infty.$$

Let $s \in (2, +\infty]$. We consider an orthogonal system $\{\varphi_n\}$ in $L_2[0, 1]$ such that

$$\|\varphi_n\|_s \leq M_n, \quad n \in \mathbb{N}, \quad (1)$$

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where $M_n \uparrow$ and $M_n \geq 1$ (see [24] or [12]). Moreover, let

$$\mu_n = \mu_n^{(s)} := \sup \left\{ \left\| \sum_{k=1}^n c_k \varphi_k \right\|_s : \sum_{k=1}^n c_k^2 = 1 \right\}, \quad \rho_n = \left(\sum_{k=n}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}. \quad (2)$$

For one variable function Marcinkiewicz and Zygmund [12] proved the following theorems.

Theorem 1.1 *Assume that the orthogonal system $\{\varphi_n\}$ satisfies the condition (1) and $2 \leq p < s$. If the real number sequence $\{a_n\}$ satisfies the condition*

$$\sum_{n=1}^{\infty} |a_n|^p M_n^{(p-2)\frac{s}{s-2}} n^{(p-2)\frac{s-1}{s-2}} < +\infty,$$

then the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

converges in L_p to some function $f \in L_p[0, 1]$ and

$$\|f\|_p \leq C_{p,s} \left(\sum_{n=1}^{\infty} |a_n|^p M_n^{(p-2)\frac{s}{s-2}} n^{(p-2)\frac{s-1}{s-2}} \right)^{\frac{1}{p}}.$$

Theorem 1.2 *Let the orthogonal system $\{\varphi_n\}$ satisfy the condition (1), and $\frac{s}{s-1} = \mu < p \leq 2$. Then the Fourier coefficients $a_n(f)$ of the function $f \in L_p[0, 1]$ with respect to the system $\{\varphi_n\}$ satisfy the inequality*

$$\left(\sum_{n=1}^{\infty} |a_n(f)|^p M_n^{(p-2)\frac{s}{s-2}} n^{(p-2)\frac{s-1}{s-2}} \right)^{\frac{1}{p}} \leq C_{p,s} \|f\|_p.$$

There are several generalizations of Theorems 1.1 and 1.2 for different function spaces and systems (see e.g. [5–7, 13] and the references therein).

In particular, Flett [5] generalized this to the case of Lorentz spaces and Maslov [13] derived generalizations of both Theorem 1.1 and Theorem 1.2 to the case of Orlicz spaces.

Some inequalities related to the summability of the Fourier coefficients in bounded orthonormal systems with functions from some Lorentz spaces were investigated e.g. by Stein [23], Bochkarev [3], Kopezhanova and Persson [9] and Kopezhanova [8].

Moreover, as a further generalization of a result of Kolyada [7] Kirillov proved an essential generalization of Theorem 1 [6].

In [2] the authors recently generalized and complemented all statements mentioned above. In particular, we proved the following generalizations.

Theorem 1.3 *Let $2 < q < s \leq +\infty$, $\alpha \in (-\infty, +\infty)$, $r > 0$ and $\delta = \frac{rs(q-2)}{q(s-2)}$. If $\{a_n\} \in l_2$ and*

$$\Omega_{q,r,\alpha} = \left\{ \sum_{n=1}^{\infty} (\rho_n^r - \rho_{n+1}^r) \mu_n^{\delta} \left(1 + \frac{2s}{s-2} \ln \mu_n \right)^{\alpha r} \right\}^{\frac{1}{r}} < +\infty,$$

where ρ_n and μ_n are defined by (2), then the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

with respect to an orthogonal system $\{\varphi_n\}_{n=1}^{\infty}$, which satisfies the condition (1), converges to some function $f \in L_{q,r}(\log L)^{\alpha}$ and $\|f\|_{q,r,\alpha} \leq C\Omega_{q,r,\alpha}$.

Remark For the case $\alpha = 0$, Theorem 1.3 coincides with Theorem 1 in [6].

Theorem 1.4 Let $s \in (2, +\infty]$, $\frac{s}{s-1} < q < 2$, $r > 1$, $\alpha \in \mathbb{R}$ and $\delta = \frac{r(q-2)s}{q(s-2)}$. If $f \in L_{q,r}(\log L)^{\alpha}$, then the inequality

$$\left[\sum_{n=1}^{\infty} \left(\sum_{k=v_n}^{v_{n+1}-1} |a_k(f)|^2 \right)^{\frac{r}{2}} (1 + \log \mu_{v_n})^{r\alpha} \mu_{v_n}^{\delta} \right]^{\frac{1}{r}} \leq C \|f\|_{q,r,\alpha}$$

holds, where μ_{v_n} are defined by (2) and $a_k(f)$ denote the Fourier coefficients of f with respect to an orthogonal system satisfying (1).

The methods of proofs of Theorems 1.3 and 1.4 presented in [2] are not sufficient to cover the case $q = 2$ in both cases. In this paper we fill in this gap by proving complements of Theorem 1 in [6] (see Theorem 2.1) and Theorem 1.2 (see Theorem 3.1) for this case. In Sect. 4 we include some concluding remarks with comparisons of other recent research of this type (see [8, 15] and [17]) and suggestions of further possibilities for development of this area.

2 A complement of Theorem 1.3. The case $q = 2$

Our main result in this section reads as follows.

Theorem 2.1 Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthogonal system, which satisfies the condition (1) and $s \in (2, +\infty]$, $0 < \theta \leq 2$, $0 \leq \alpha < +\infty$. If $\{a_n\} \in l_2$ and

$$\Lambda_{2,\theta,\alpha}(a) = \sum_{n=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^n M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} (\rho_n^{\theta} - \rho_{n+1}^{\theta}) < +\infty,$$

then the series $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ converges in the space $L_{2,\theta}(\log L)^{\alpha}$ to some function $f \in L_{2,\theta}(\log L)^{\alpha}$ and $\|f\|_{2,\theta,\alpha} \leq C(\Lambda_{2,\theta,\alpha})^{1/\theta}$.

For the proof of this theorem we need the following well-known results of Kolyada [7].

Lemma 2.2 Let $\sum_{k=1}^{\infty} \alpha_k$ be a convergent numerical series and $0 < s < q < \infty$. Then

$$\left| \sum_{k=1}^{\infty} \alpha_k \right|^q \leq 2^q \sup_n \left| \sum_{k=1}^n \alpha_k \right|^{q-s} \left| \sum_{k=n}^{\infty} \alpha_k \right|^s.$$

Lemma 2.3 Let $\alpha_n \geq 0$ and $\varepsilon_n > 0$, and assume that for some $\beta \in (0, 1)$ $\varepsilon_{n+1} \leq \beta \varepsilon_n$ ($n = 1, 2, \dots$). Then the following inequalities hold for all $r > 0$:

$$\sum_{n=1}^{\infty} \varepsilon_n \left(\sum_{k=1}^n \alpha_k \right)^r \leq C_{r,\beta} \sum_{n=1}^{\infty} \varepsilon_n \alpha_n^r,$$

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \alpha_k \right)^r \varepsilon_n^{-1} \leq C_{r,\beta} \sum_{n=1}^{\infty} \alpha_n^r \varepsilon_n^{-1}.$$

Here, as usual, $C_{r,\beta}$ denotes a positive constant depending only on r and β .

Proof of Theorem 2.1 Since $\rho_n \downarrow 0, n \rightarrow +\infty$, we can choose a sequence of natural numbers $\{v_k\}_{k=1}^{\infty}$ as follows: $v_1 = 1$ and $v_{k+1} = \min\{n \in \mathbb{N} : \rho_n \leq \frac{1}{2}\rho_{v_k}\}$, where ρ_n are defined by (2). Then

$$\rho_{v_{k+1}} \leq \frac{1}{2}\rho_{v_k}, \quad \rho_{v_{k+1}-1} > \frac{1}{2}\rho_{v_k}. \quad (3)$$

We define $u_k(x) := |\sum_{n=v_k}^{v_{k+1}-1} a_n \varphi_n(x)|$. Then, by using Parseval's relation, we have

$$\|u_k\|_2 = \left(\sum_{n=v_k}^{v_{k+1}-1} |a_n|^2 \right)^{\frac{1}{2}} \leq \rho_{v_k} := \varepsilon_k. \quad (4)$$

For each number $l \in \mathbb{N}$ we consider the function $f_l(x) := \sum_{n=1}^{v_{l+1}-1} a_n \varphi_n(x)$. Next we show that $\{f_l\}$ is a fundamental sequence in the space $L_{2,\theta}(\log L)^\alpha$. For any natural numbers m, l by the property of the modulus of numbers we have

$$|f_l(x) - f_m(x)| \leq \sum_{k=m}^l u_k(x).$$

By using Lemma 2.2 with $q = 1$ we get

$$|f_l(x) - f_{m-1}(x)| \leq 2 \sup_{n=m, \dots, l} \left| \sum_{k=m}^n u_k(x) \right|^{1-\beta} \left| \sum_{k=n}^l u_k(x) \right|^\beta, \quad (5)$$

where the number $\beta \in (0, 1]$ will be chosen later on in the proof. By the property of non-increasing rearrangement of a function (see e.g. [10], p. 89) we know that

$$f^*(t) \leq \frac{1}{t} \int_0^t f^*(y) dy = \frac{1}{t} \sup_{\substack{e \subset [0,1] \\ |e|=t}} \int_e |f(x)| dx. \quad (6)$$

Now, by using (5) and (6) we can conclude that

$$(f_l - f_{m-1})^*(t) \leq 2 \sup_{n=m, \dots, l} \left[\frac{1}{t} \sup_{\substack{e \subset [0,1] \\ |e|=t}} \int_e \left| \sum_{k=m}^n u_k(x) \right|^{1-\beta} \left| \sum_{k=n}^l u_k(x) \right|^\beta dx \right]. \quad (7)$$

Next we use Hölder's inequality with exponents $p = \frac{1}{\beta} > 1$ and $p' = 1/(1 - \beta)$ in (7) and find that

$$\begin{aligned} (f_l - f_{m-1})^*(t) &\leq 2 \sup_{n=m, \dots, l} \left[\frac{1}{t} \sup_{\substack{e \in [0,1] \\ |e|=t}} \left(\int_e \left| \sum_{k=m}^n u_k(x) \right| dx \right)^{1-\beta} \left(\int_e \left| \sum_{k=n}^l u_k(x) \right| dx \right)^{\beta} \right] \\ &= 2 \sup_{n=m, \dots, l} \left[\left(\frac{1}{t} \int_0^t \left(\sum_{k=m}^n u_k(y) \right)^* dy \right)^{1-\beta} \left(\frac{1}{t} \int_0^t \left(\sum_{k=n}^l u_k(y) \right)^* dy \right)^{\beta} \right]. \end{aligned}$$

We raise both sides to the power θ , multiply by $\frac{\theta}{2}(1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1}$ and integrate. Then

$$\begin{aligned} &\frac{\theta}{2} \int_0^1 [(f_l - f_{m-1})^*(t)]^{\theta} (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \\ &\leq 2^{\theta} \frac{\theta}{2} \sum_{n=m}^l \int_0^1 \left[\frac{1}{t} \int_0^t \left(\sum_{k=m}^n u_k(y) \right)^* dy \right]^{\theta(1-\beta)} \\ &\quad \times \left[\frac{1}{t} \int_0^t \left(\sum_{k=n}^l u_k(y) \right)^* dy \right]^{\theta\beta} (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \\ &= 2^{\theta} \frac{\theta}{2} \sum_{n=m}^l \int_0^1 t^{\frac{\theta}{2}-1} (1 + |\ln t|)^{\alpha\theta} F_{m,n}^{\theta(1-\beta)}(t) \Phi_{n,l}^{\theta\beta}(t) dt. \end{aligned} \quad (8)$$

For simplicity we introduce the notations

$$F_{m,n}(t) = \frac{1}{t} \int_0^t \left(\sum_{k=m}^n u_k(y) \right)^* dy \quad \text{and} \quad \Phi_{n,l}(t) = \frac{1}{t} \int_0^t \left(\sum_{k=n}^l u_k(y) \right)^* dy.$$

Choose the number r such that $1 < r < \theta \leq 2$ and note that $s := \frac{2(\theta-r)}{2-r} > 0$ and $\beta = \frac{s}{\theta}$. Then $\beta\theta = s$ and $\frac{(r-2)(\theta-s)}{2r} = \frac{\theta}{2} - 1$. Therefore,

$$\int_0^1 F_{m,n}^{\theta(1-\beta)}(t) \Phi_{n,l}^{\theta\beta}(t) (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt = \int_0^1 F_{m,n}^{\theta-s}(t) \Phi_{n,l}^s(t) (1 + |\ln t|)^{\alpha\theta} t^{\frac{(r-2)(\theta-s)}{2r}} dt. \quad (9)$$

By again using the Hölder inequality now with exponents $p = \frac{r}{\theta-s}$ and $p' = \frac{r}{r-\theta+s}$ on the integral on the right hand side of (9) we find that

$$\begin{aligned} &\int_0^1 F_{m,n}^{\theta(1-\beta)}(t) \Phi_{n,l}^{\theta\beta}(t) (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \\ &\leq \left\{ \int_0^1 F_{m,n}^r(t) (1 + |\ln t|)^{\alpha\theta v} t^{\frac{r}{2}-1} dt \right\}^{\frac{\theta-s}{r}} \left\{ \int_0^1 \Phi_{n,l}^{sv'}(t) dt \right\}^{\frac{1}{v'}} \\ &\leq C \left\| \frac{1}{t} \int_0^t \left(\sum_{k=m}^n u_k(y) \right)^* dy \right\|_{2,r, \frac{\alpha\theta v}{r}}^{\theta-s} \left\| \sum_{k=n}^l u_k \right\|_2^s. \end{aligned} \quad (10)$$

Next, by using the norm property of l_2 spaces, the Parseval theorem and the definition of the numbers v_k , we obtain (see (4))

$$\left\| \sum_{k=n}^l u_k \right\|_2 \leq \sum_{k=n}^l \|u_k\|_2 = \sum_{k=n}^l \left(\sum_{j=v_k}^{v_{k+1}-1} a_j^2 \right)^{\frac{1}{2}} \leq \sum_{k=n}^l \rho_{v_k} \leq 2\rho_{v_n}.$$

Therefore, from inequality (10) it follows that

$$\begin{aligned} & \int_0^1 F_{m,n}^{\theta(1-\beta)}(t) \Phi_{n,l}^{\theta\beta}(t) (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \\ & \leq \left\| \frac{1}{t} \int_0^t \left(\sum_{k=m}^n u_k(y) \right)^* dy \right\|_{2,r, \frac{\alpha\theta v}{r}}^{\theta-s} \rho_{v_n}^s \leq C \left(\sum_{k=m}^n \left\| \frac{1}{t} \int_0^t u_k^*(y) dy \right\|_{2,r, \frac{\alpha\theta v}{r}} \right)^{\theta-s} \rho_{v_n}^s. \end{aligned} \quad (11)$$

By applying Lemma 2.3 and inequalities (8) and (11) we obtain

$$\begin{aligned} & \int_0^1 [(f_l - f_{m-1})^*(t)]^\theta (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \\ & \leq C \sum_{n=m}^l \left(\sum_{k=m}^n \|u_k\|_{2,r, \frac{\alpha\theta v}{r}} \right)^{\theta-s} \rho_{v_n}^s \leq C \sum_{n=m}^l \|u_n\|_{2,r, \frac{\alpha\theta v}{r}}^{\theta-s} \rho_{v_n}^s. \end{aligned} \quad (12)$$

Since $1 < r < 2$, then, by the inequality of different metrics (see [1]), we get

$$\|u_n\|_{2,r, \frac{\alpha\theta v}{r}} \leq C \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{\frac{1}{r} - \frac{1}{2} + \frac{\alpha\theta v}{r}} \|u_n\|_2.$$

Therefore taking into account that $(\frac{1}{r} - \frac{1}{2} + \frac{\alpha\theta v}{r})(\theta - s) = 1 - \frac{\theta}{2} + \alpha\theta$ we obtain

$$\begin{aligned} \|u_n\|_{2,r, \frac{\alpha\theta v}{r}}^{\theta-s} & \leq C \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{(\frac{1}{r} - \frac{1}{2} + \frac{\alpha\theta v}{r})(\theta-s)} \|u_n\|_2^{\theta-s} \\ & = C \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1 - \frac{\theta}{2} + \alpha\theta} \|u_n\|_2^{\theta-s} \\ & \leq C \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1 - \frac{\theta}{2} + \alpha\theta} \rho_{v_n}^{\theta-s}. \end{aligned}$$

Hence, from (12) it follows that

$$\int_0^1 [(f_l - f_m)^*(t)]^\theta (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \leq C \sum_{n=m}^l \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1 - \frac{\theta}{2} + \alpha\theta} \rho_{v_n}^\theta. \quad (13)$$

By definition of the numbers v_n (see (3)) we have $\rho_{v_n} < 2\rho_{v_{n+1}-1}$ and $\rho_{v_{n+2}} \leq \frac{1}{2}\rho_{v_{n+1}} < \frac{1}{2}\rho_{v_{n+1}-1}$. Thus $\rho_{v_{n+1}-1}^\theta - \rho_{v_{n+2}}^\theta \geq (1 - 1/2)^\theta \rho_{v_{n+1}-1}^\theta$ so that

$$\rho_{v_{n+1}-1}^\theta \leq \frac{2^\theta}{2^\theta - 1} (\rho_{v_{n+1}-1}^\theta - \rho_{v_{n+2}}^\theta).$$

Therefore,

$$\begin{aligned}
& \sum_{n=m}^l \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \rho_{v_n}^\theta \\
& \leq 2^\theta \sum_{n=m}^l \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \rho_{v_{n+1}-1}^\theta \\
& \leq 2^\theta \frac{2^\theta}{2^\theta-1} \sum_{n=m}^l \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} (\rho_{v_{n+1}-1}^\theta - \rho_{v_{n+2}}^\theta) \\
& \leq \frac{2^{2\theta}}{2^\theta-1} \sum_{n=m}^l \sum_{k=v_{n+1}-1}^{v_{n+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \\
& = \frac{2^{2\theta}}{2^\theta-1} \sum_{n=m}^l \left[(\rho_{v_{n+1}-1}^\theta - \rho_{v_{n+1}}^\theta) \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \right. \\
& \quad \left. + \sum_{k=v_{n+1}}^{v_{n+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \right] \\
& \leq \frac{2^{2\theta}}{2^\theta-1} \sum_{n=m}^l \left[\sum_{k=v_{n+1}}^{v_{n+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \right. \\
& \quad \left. + \sum_{k=v_{n+1}}^{v_{n+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \right] \\
& \leq 2 \frac{2^{2\theta}}{2^\theta-1} \sum_{n=v_m}^{v_{l+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
& \sum_{n=m}^l \left(\ln \left(1 + \sum_{j=1}^{v_{n+1}-1} M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \rho_{v_n}^\theta \\
& \leq 2 \frac{2^{2\theta}}{2^\theta-1} \sum_{n=v_m}^{v_{l+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta}.
\end{aligned}$$

Hence, from (13) it follows that

$$\begin{aligned}
& \int_0^1 [(f_l - f_m)^*(t)]^\theta (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \\
& \leq C \sum_{n=v_m}^{v_{l+2}-1} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta}.
\end{aligned} \tag{14}$$

We use the assumptions in the theorem and conclude that the sequence $\{f_l\} \subset L_{2,\theta}(\log L)^\alpha$ is fundamental in the space $L_{2,\theta}(\log L)^\alpha$. Hence, since the space $L_{2,\theta}(\log L)^\alpha$ is complete (see

[21]) there exists a function $f \in L_{2,\theta}(\log L)^\alpha$ such that $\|f - f_l\|_{2,\theta,\alpha} \rightarrow 0$ for $l \rightarrow \infty$ and

$$f(x) \sim \sum_{n=1}^{\infty} a_n \varphi_n(x).$$

By now taking the limit $l \rightarrow \infty$ in (14) we get

$$\int_0^1 [(f - f_m)^*(t)]^\theta (1 + |\ln t|)^{\alpha\theta} t^{\frac{\theta}{2}-1} dt \leq C \sum_{n=v_m}^{\infty} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta}.$$

Finally, in this inequality we put $m = 1$ and use the norm property to conclude that

$$\|f\|_{2,\theta,\alpha} \leq C \left\{ \sum_{k=1}^{\infty} (\rho_k^\theta - \rho_{k+1}^\theta) \left(\ln \left(1 + \sum_{j=1}^k M_j^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \right\}^{\frac{1}{\theta}}.$$

The proof is complete. \square

3 A complement of Theorem 1.4. The case $q = 2$

Our main result in this section reads as follows.

Theorem 3.1 *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthogonal system, which satisfies the condition (1), $s \in (2, +\infty]$, $2 < \theta < +\infty$ and $\alpha < 0$. If the function $f \in L_{2,\theta}(\log L)^\alpha$, then*

$$\left\{ \sum_{n=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{n+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{k=v_n}^{v_{n+1}-1} a_k^2(f) \right)^{\frac{\theta}{2}} \right\}^{\frac{1}{\theta}} \leq C \|f\|_{2,\theta,\alpha},$$

where $a_k(f)$ as usual denote the Fourier coefficients with respect to the system $\{\varphi_n\}_1^\infty$.

Proof It is well known that for any function $f \in L_{q,\theta}(\log L)^\alpha$ the following relation holds (see e.g. [21]):

$$\|f\|_{q,\theta,\alpha} \asymp \sup_{\substack{g \in L_{q',\theta'}(\log L)^{-\alpha} \\ \|g\|_{q',\theta',-\alpha} \leq 1}} \left| \int_0^1 f(x)g(x) dx \right|, \quad 1/q + 1/q' = 1, \quad 1/\theta + 1/\theta' = 1. \quad (15)$$

Consider the function $g(x)$ with Fourier coefficients

$$\begin{aligned} b_n(g) &= \left\{ \sum_{k=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \right)^{\frac{\theta}{2}} \right\}^{-\frac{1}{\theta'}} \\ &\quad \times \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \right)^{\frac{\theta-2}{2}} a_n(f), \end{aligned}$$

for $n = v_k, \dots, v_{k+1} - 1$, $k \in \mathbb{N}$.

Since $\{\varphi_n\}$ is an orthogonal system and by the definition of the coefficients $b_n(g)$ we have

$$\begin{aligned} \int_0^1 f(x)g(x) dx &= \left\{ \sum_{n=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{n+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{k=v_n}^{v_{n+1}-1} a_k^2(f) \right)^{\frac{\theta}{2}} \right\}^{-\frac{1}{\theta'}} \\ &\quad \times \sum_{k=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \right)^{\frac{\theta-2}{2}} \sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \\ &= \left\{ \sum_{n=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{n+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{k=v_n}^{v_{n+1}-1} a_k^2(f) \right)^{\frac{\theta}{2}} \right\}^{-\frac{1}{\theta'}}. \end{aligned} \quad (16)$$

Hence, according to Theorem 2.1, we find that

$$\begin{aligned} \|g\|_{2,\theta',-\alpha} &\leq C \left\{ \sum_{n=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^n M_l^2 \right) \right)^{1-\frac{\theta'}{2}-\alpha\theta'} (\rho_n^{\theta'} - \rho_{n+1}^{\theta'}) \right\}^{\frac{1}{\theta'}} \\ &\leq C \left\{ \sum_{k=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta'}{2}-\alpha\theta'} (\rho_{v_k}^{\theta'} - \rho_{v_{k+1}}^{\theta'}) \right\}^{\frac{1}{\theta'}}, \end{aligned}$$

where $\rho_n = (\sum_{l=n}^{\infty} |b_l(g)|^2)^{1/2}$, $n \in \mathbb{N}$ and $-\alpha > 0$.

If $a > b > 0$, $0 < \beta \leq 1$, then $a^\beta - b^\beta \leq (a-b)^\beta$. Since $\theta'/2 < 1$, by this inequality we obtain

$$\begin{aligned} \rho_{v_k}^{\theta'} - \rho_{v_{k+1}}^{\theta'} &= \left(\sum_{l=v_k}^{\infty} |b_l(g)|^2 \right)^{\theta'/2} - \left(\sum_{l=v_{k+1}}^{\infty} |b_l(g)|^2 \right)^{\theta'/2} \\ &\leq \left(\sum_{l=v_k}^{\infty} |b_l(g)|^2 - \sum_{l=v_{k+1}}^{\infty} |b_l(g)|^2 \right)^{\theta'/2} = \left(\sum_{l=v_k}^{v_{k+1}-1} |b_l(g)|^2 \right)^{\frac{\theta'}{2}}. \end{aligned}$$

Therefore,

$$\|g\|_{2,\theta',-\alpha} \leq C \left\{ \sum_{k=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta'}{2}-\alpha\theta'} \left(\sum_{l=v_k}^{v_{k+1}-1} |b_l(g)|^2 \right)^{\frac{\theta'}{2}} \right\}^{\frac{1}{\theta'}}.$$

By again using the definition of the coefficients $b_n(g)$ we obtain

$$\begin{aligned} \left(\sum_{n=v_k}^{v_{k+1}-1} b_n^2(g) \right)^{\frac{1}{2}} &= \left\{ \sum_{k=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \right)^{\frac{\theta}{2}} \right\}^{-\frac{1}{\theta'}} \\ &\quad \times \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \right)^{\frac{\theta-1}{2}}. \end{aligned}$$

Then $\|g\|_{2,\theta',-\alpha} \leq C$. Thus, the function $g_0 := C^{-1}g \in L_{2,\theta'}(\log L)^{-\alpha}$ and $\|g_0\|_{2,\theta',-\alpha} \leq 1$.

Hence, by using (15), from (16) it follows that

$$\|f\|_{2,\theta,\alpha} \geq \left| \int_0^1 f(x)g_0(x) dx \right|$$

$$\geq C^{-1} \left\{ \sum_{k=1}^{\infty} \left(\ln \left(1 + \sum_{l=1}^{v_{k+1}-1} M_l^2 \right) \right)^{1-\frac{\theta}{2}+\alpha\theta} \left(\sum_{n=v_k}^{v_{k+1}-1} a_n^2(f) \right)^{\frac{\theta}{2}} \right\}^{\frac{1}{\theta}}.$$

The proof is complete. \square

4 Concluding remarks

We say that a function f on $(0, 1)$ or $(0, \infty)$ is quasi-increasing (quasi-decreasing) if, for all $x \leq y$ and some $C > 0$, $f(x) \leq Cf(y)$ ($f(y) \leq Cf(x)$). Moreover, we say that a positive function on (a, b) , $0 \leq a < b < \infty$, is a quasi-monotone weight if $\lambda(x)x^c$ is quasi-increasing or quasi-decreasing for some $c \in \mathbb{R}$. It is then natural to define the more general Lorentz spaces $\Lambda_q(\lambda)$ than the usual one $L_{p,q}$ where $\lambda(t) = t^{1/p}$. In particular, if $\lambda(t) = (1 + |\ln t|)^\alpha$, $0 < t \leq 1$, $\lambda(t) = 0$, $t \geq 1$, then the spaces $\Lambda_q(\lambda)$ and $L_{p,q}(\log L)^\alpha$ coincide.

Remark 4.1 In Refs. [9] and [8] these more general Lorentz spaces $\Lambda_q(\lambda)$ were defined and investigated in a similar way but only for bounded systems. Here $\lambda(t)$ is a quasi-monotone weight considered early in Ref. [16] by Persson but then used only for Fourier inequalities related to the trigonometric system.

Remark 4.2 Quasi-monotone weights are very useful and possible to handle in various situations in analysis since we have good control of the growth both up and down as $t \rightarrow 0$ or $t \rightarrow \infty$. For example the method of “interpolation with a parameter function” heavily depends on this idea (see [18]). The close relation to Matuszewska–Orlicz indices, the Bari–Stechkin condition and other remarkable properties were investigated in [19].

Remark 4.3 In [8] (see Theorem 2.1, Theorem 2.3), theorems on the convergence of series of the Fourier coefficients of a function from the generalized Lorentz space $\Lambda_q(\lambda)$ with respect to regular systems are proved. It is well known that a regular system is uniformly bounded (see [15], p. 117). Therefore, the assertions of Theorem 2.1 and Theorem 3.1 of this paper cannot follow from the results of [8].

Remark 4.4 For the type of problems considered in this paper and [2] it is natural to consider the following slight generalizations of the classes A and B considered in [8] and [17]: $A^* = \bigcup_{s>0} A_\delta$ and $B^* = \bigcup_{s>0} B_\delta$, where A_δ consists of positive functions $\lambda(t)$ such that $\lambda(t)t^{-\delta}$ is quasi-increasing and $\lambda(t)t^{-(1/2-\delta)}$ quasi-decreasing and B_δ consists of positive functions $\omega(t)$ such that $\omega(t)t^{-1/2-\delta}$ is quasi-increasing and $\omega(t)t^{-1+\delta}$ is quasi-decreasing.

Example 4.5 It is well known that any concave function $\psi(t)$ is quasi-monotone. More exactly, $\psi(t)$ is nondecreasing and $\psi(t)/t$ is nonincreasing. A simple proof can be found on page 142 Ref. [11].

Inspired by the discussions above and in order to be able to compare with a recent result of Doktorski [4] we introduce the generalized Lorentz space $L_{\psi,\theta}$ as follows: For $\psi(t)$ quasi-monotone and $\theta > 0$ we say that the measurable functions $f \in L_{\psi,\theta}$ whenever

$$\|f\|_{\psi,\theta} = \left(\int_0^1 f^{*\theta}(t) \psi^\theta(t) \frac{dt}{t} \right)^{\frac{1}{\theta}} < \infty.$$

For the function ψ we set

$$\alpha_\psi = \lim_{t \rightarrow 0} \frac{\psi(2t)}{\psi(t)}, \quad \beta_\psi = \overline{\lim}_{t \rightarrow 0} \frac{\psi(2t)}{\psi(t)}.$$

It is well known that $1 \leq \alpha_\psi \leq \beta_\psi \leq 2$ (see e.g. [20]).

Consider the set of all non-negative functions on $[0, 1]$, ψ for which $(\log 2/t)^\varepsilon \psi(t) \uparrow +\infty$ and $(\log 2/t)^{-\varepsilon} \psi(t) \downarrow 0$ for $t \downarrow 0$, $\forall \varepsilon > 0$ (cf. [22]) and this set is also denoted by SVL.

By making modifications of the proof of Theorem 2.1 it is possible to prove the following generalization of this theorem.

Theorem 4.6 *Let $\{\varphi_n\}_{n=1}^\infty$ be an orthogonal system, which satisfies the condition (1) and $s \in (2, +\infty]$. Moreover, assume that ψ is a quasi-monotone function, which satisfy the conditions $\alpha_\psi = \beta_\psi = 2^{1/2}$,*

$$\sup_{t \in (0,1]} \frac{t^{1/2}}{\psi(t)} < \infty$$

and $\frac{t^{1/2}}{\psi(t)} \in \text{SVL}$.

If $1 < \theta \leq 2$, $\{a_n\} \in l_2$ and

$$\Lambda_{\psi,\theta}(a) = \sum_{n=1}^{\infty} \left(\frac{\psi((1 + \sum_{l=1}^n M_l^2)^{-1})}{(1 + \sum_{l=1}^n M_l^2)^{-1/2}} \right)^\theta \left(\ln \left(1 + \sum_{l=1}^n M_l^2 \right) \right)^{(\frac{1}{\theta} - \frac{1}{2})\theta} (\rho_n^\theta - \rho_{n+1}^\theta) < +\infty,$$

then the series $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ converges in the space $L_{\psi,\theta}$ to some function $f \in L_{\psi,\theta}$ and $\|f\|_{\psi,\theta} \leq C(\Lambda_{\psi,\theta})^{1/\theta}$.

Remark 4.7 In the case $\psi(t) = t^{1/2}(1 + \ln |t|)^\alpha$ Theorem 4.6 implies Theorem 2.1.

Remark 4.8 For a uniformly bounded system $\{\varphi_n\}$, Theorem 4.6 was recently proved different way and in a slightly different form by Doktorski [4].

Remark 4.9 The remarks above open the possibility of generalizing and unifying all the results in [2, 4, 8, 9] and this paper. The present authors hope to investigate this in a forthcoming paper.

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